

# Area-Preserving Parameterizations for Spherical Ellipses

## supplemental document

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### 1. Numerical Validation

In Table [S1](#) we provide numerical validation of the expressions for the spherical ellipse's solid angle  $\Omega_D$  in Equations (11) and (19) based on elliptical integrals. We validate our results against previously published expressions for the disk solid angle from Paxton [[Pax59](#)] and Conway [[Con10](#)]. We also integrate numerically our Equations (6) and (16) for further numerical evidence.

For the numerical tests, we consider a disk with radius 2 at distance of 3 from the origin. We also consider different rotations of the disk over its center, so it forms an angle  $\theta$  with the direction to the origin. Specifically, we consider the rotation from 0 to  $\pi/2$  radians, since due to symmetry all the other possible rotations are equivalent. The computed values are collected in Table [S1](#), where it can be observed that all the numerical values agree.

### 2. Additional Image Comparisons

We also provide additional images, rendered in Mitsuba [[Jak10](#)] with global illumination and next-event estimation, showing both surface transport (Figure [S1](#)) and complete volume and surface transport (Figure [S2](#)). We include comparisons of our method against traditional surface area sampling. The results demonstrate that the overhead of our method pays off by providing noticeably lower-noise results.

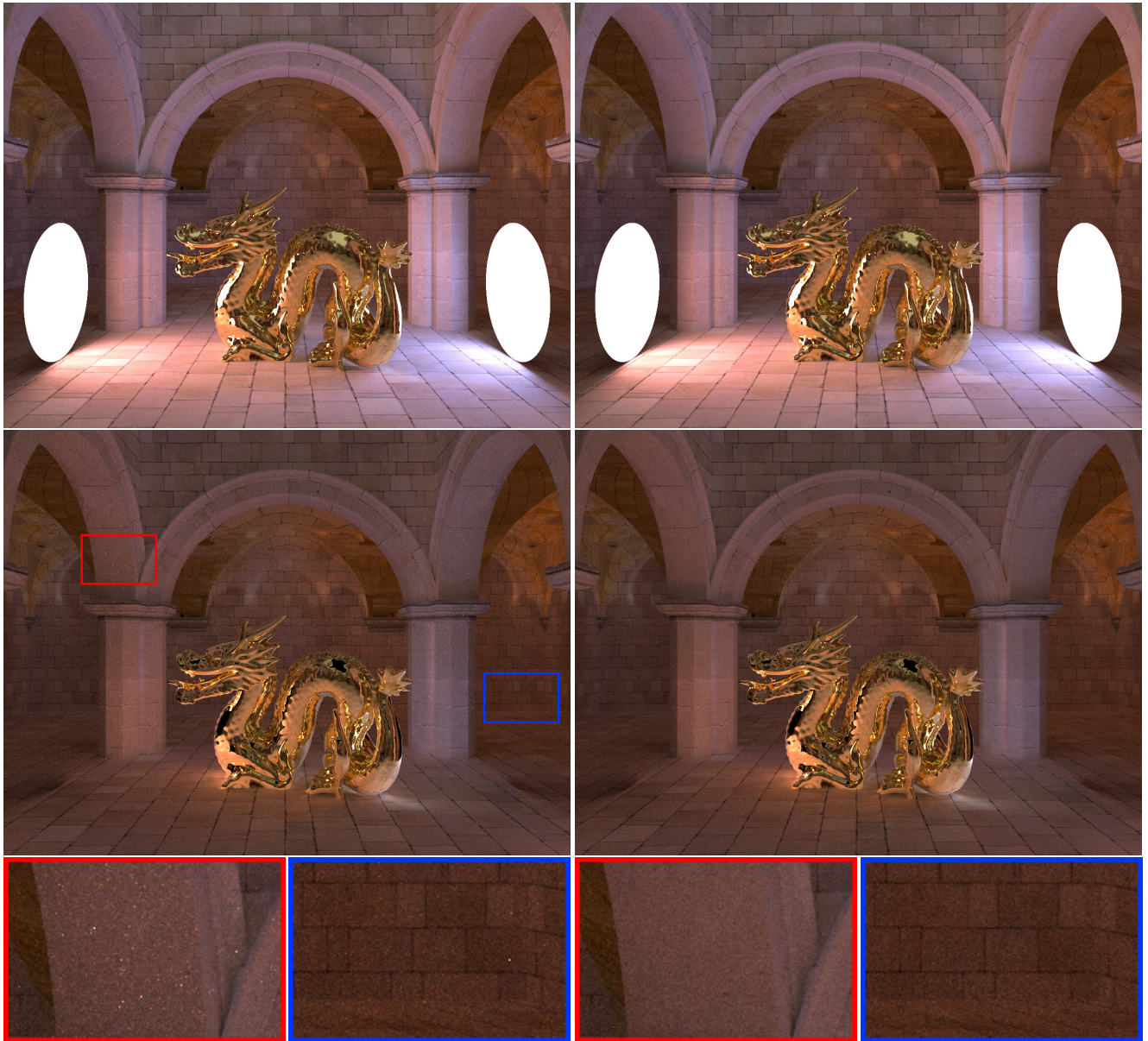
**Table S1:** Disk solid angle  $\Omega_D$  calculated for different values of  $\theta$  with each of the considered methods.

$\theta$	[Pax59]	[Con10]	Eq. (6)	Eq. (11)	Eq. (16)	Eq. (19)
0.0155524389	1.0552516368	1.0552516368	1.0552516368	1.0552516368	1.0552516368	1.0552516368
0.0311048778	1.0552291322	1.0552291322	1.0552291322	1.0552291322	1.0552291322	1.0552291322
0.0466573166	1.0551914715	1.0551914715	1.0551914715	1.0551914715	1.0551914715	1.0551914715
0.0622097555	1.0551384242	1.0551384242	1.0551384242	1.0551384242	1.0551384242	1.0551384242
0.0777621944	1.0550696672	1.0550696672	1.0550696672	1.0550696672	1.0550696672	1.0550696672
0.0933146333	1.0549847831	1.0549847831	1.0549847831	1.0549847831	1.0549847831	1.0549847831
0.1088670722	1.0548832597	1.0548832597	1.0548832597	1.0548832597	1.0548832597	1.0548832597
0.1244195110	1.0547644877	1.0547644877	1.0547644877	1.0547644877	1.0547644877	1.0547644877
0.1399719499	1.0546277599	1.0546277599	1.0546277599	1.0546277599	1.0546277599	1.0546277599
0.1555243888	1.0544722683	1.0544722683	1.0544722683	1.0544722683	1.0544722683	1.0544722683
0.1710768277	1.0542971023	1.0542971023	1.0542971023	1.0542971023	1.0542971023	1.0542971023
0.1866292665	1.0541012459	1.0541012459	1.0541012459	1.0541012459	1.0541012459	1.0541012459
0.2021817054	1.0538835747	1.0538835747	1.0538835747	1.0538835747	1.0538835747	1.0538835747
0.2177341443	1.0536428523	1.0536428523	1.0536428523	1.0536428523	1.0536428523	1.0536428523
0.2332865832	1.0533777275	1.0533777275	1.0533777275	1.0533777275	1.0533777275	1.0533777275
0.2488390221	1.0530867294	1.0530867294	1.0530867294	1.0530867294	1.0530867294	1.0530867294
0.2643914609	1.0527682637	1.0527682637	1.0527682637	1.0527682637	1.0527682637	1.0527682637
0.2799438998	1.0524206073	1.0524206073	1.0524206073	1.0524206073	1.0524206073	1.0524206073
0.2954963387	1.0520419040	1.0520419040	1.0520419040	1.0520419040	1.0520419040	1.0520419040
0.3110487776	1.0516301581	1.0516301581	1.0516301581	1.0516301581	1.0516301581	1.0516301581
0.3266012165	1.0511832287	1.0511832287	1.0511832287	1.0511832287	1.0511832287	1.0511832287
0.3421536553	1.0506988233	1.0506988233	1.0506988233	1.0506988233	1.0506988233	1.0506988233
0.3577060942	1.0501744907	1.0501744907	1.0501744907	1.0501744907	1.0501744907	1.0501744907
0.3732585331	1.0496076133	1.0496076133	1.0496076133	1.0496076133	1.0496076133	1.0496076133
0.3888109720	1.0489953992	1.0489953992	1.0489953992	1.0489953992	1.0489953992	1.0489953992
0.4043634109	1.0483348733	1.0483348733	1.0483348733	1.0483348733	1.0483348733	1.0483348733
0.4199158497	1.0476228681	1.0476228681	1.0476228681	1.0476228681	1.0476228681	1.0476228681
0.4354682886	1.0468560138	1.0468560138	1.0468560138	1.0468560138	1.0468560138	1.0468560138
0.4510207275	1.0460307276	1.0460307276	1.0460307276	1.0460307276	1.0460307276	1.0460307276
0.4665731664	1.0451432018	1.0451432018	1.0451432018	1.0451432018	1.0451432018	1.0451432018
0.4821256053	1.0441893921	1.0441893921	1.0441893921	1.0441893921	1.0441893921	1.0441893921
0.4976780441	1.0431650046	1.0431650046	1.0431650046	1.0431650046	1.0431650046	1.0431650046
0.5132304830	1.0420654810	1.0420654810	1.0420654810	1.0420654810	1.0420654810	1.0420654810
0.5287829219	1.0408859847	1.0408859847	1.0408859847	1.0408859847	1.0408859847	1.0408859847
0.5443353608	1.0396213842	1.0396213842	1.0396213842	1.0396213842	1.0396213842	1.0396213842
0.5598877996	1.0382662364	1.0382662364	1.0382662364	1.0382662364	1.0382662364	1.0382662364
0.5754402385	1.0368147682	1.0368147682	1.0368147682	1.0368147682	1.0368147682	1.0368147682
0.5909926774	1.0352608576	1.0352608576	1.0352608576	1.0352608576	1.0352608576	1.0352608576
0.6065451163	1.0335980126	1.0335980126	1.0335980126	1.0335980126	1.0335980126	1.0335980126
0.6220975552	1.0318193499	1.0318193499	1.0318193499	1.0318193499	1.0318193499	1.0318193499
0.6376499940	1.0299175710	1.0299175710	1.0299175710	1.0299175710	1.0299175710	1.0299175710
0.6532024329	1.0278849376	1.0278849376	1.0278849376	1.0278849376	1.0278849376	1.0278849376
0.6687548718	1.0257132454	1.0257132454	1.0257132454	1.0257132454	1.0257132454	1.0257132454
0.6843073107	1.0233937958	1.0233937958	1.0233937958	1.0233937958	1.0233937958	1.0233937958
0.6998597496	1.0209173665	1.0209173665	1.0209173665	1.0209173665	1.0209173665	1.0209173665
0.7154121884	1.0182741796	1.0182741796	1.0182741796	1.0182741796	1.0182741796	1.0182741796
0.7309646273	1.0154538687	1.0154538687	1.0154538687	1.0154538687	1.0154538687	1.0154538687
0.7465170662	1.0124454434	1.0124454434	1.0124454434	1.0124454434	1.0124454434	1.0124454434
0.7620695051	1.0092372521	1.0092372521	1.0092372521	1.0092372521	1.0092372521	1.0092372521
0.7776219440	1.0058169428	1.0058169428	1.0058169428	1.0058169428	1.0058169428	1.0058169428
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0.9331463327	0.9566408591	0.9566408591	0.9566408591	0.9566408591	0.9566408591	0.9566408591
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1.0886707215	0.8656720378	0.8656720378	0.8656720378	0.8656720378	0.8656720378	0.8656720378
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1.4463768158	0.3212275309	0.3212275309	0.3212275309	0.3212275309	0.3212275309	0.3212275309
1.4619292546	0.2835614167	0.2835614167	0.2835614167	0.2835614167	0.2835614167	0.2835614167
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## References

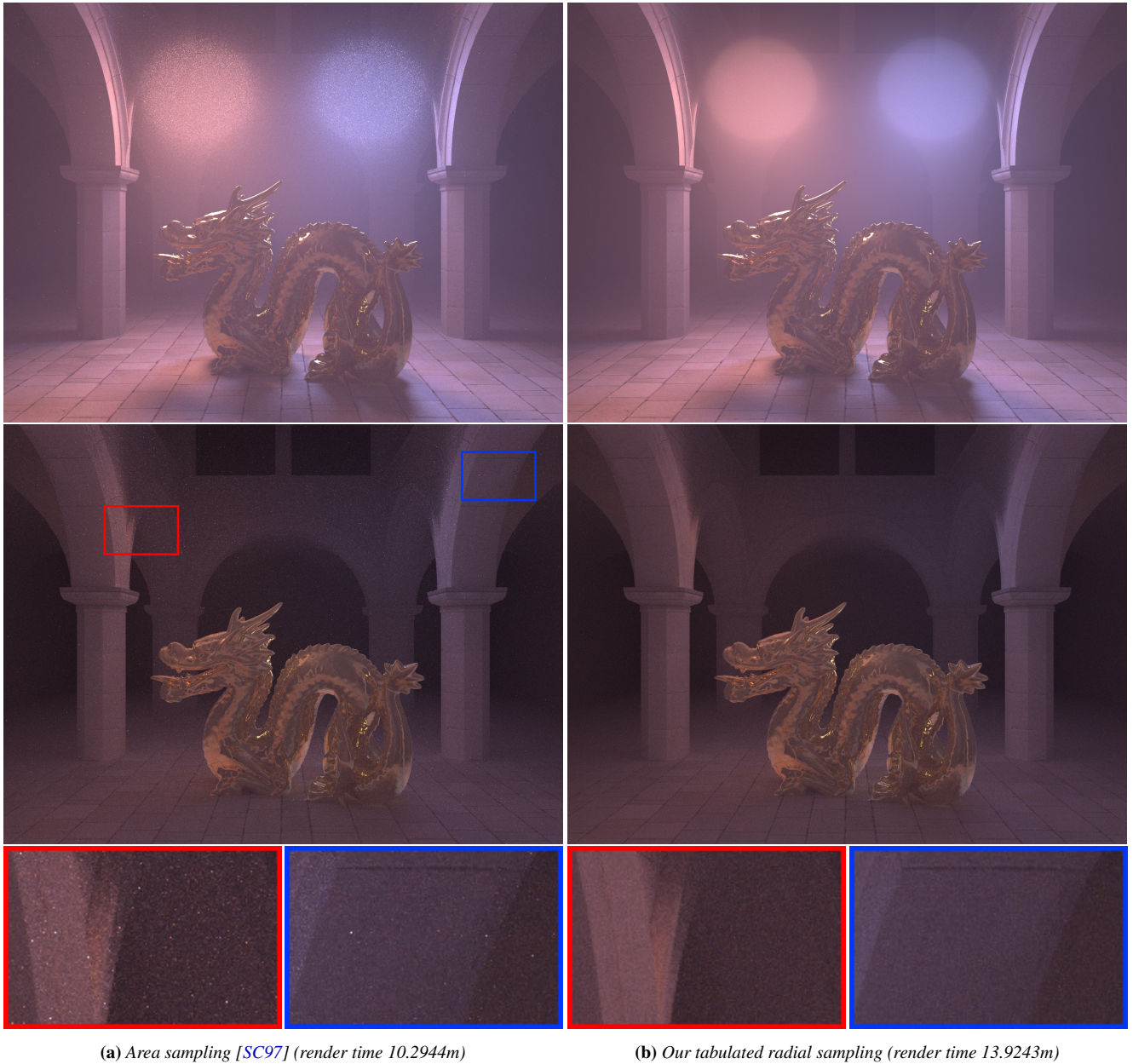
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(a) Area sampling [SC97] (render time 7.3721m)

(b) Our tabulated radial sampling (render time 9.4796m)

**Figure S1:** Top: A illuminated by two single-sided disk lights (invisible to camera rays), rendered with global illumination at  $1280 \times 960$  resolution with 512 samples/pixel. Middle: Just the indirect illumination component. Bottom: Zoom-ins into the indirect component. Even with this high sample count, uniform area sampling produces a noisy image with fireflies in the indirect illumination.



**Figure S2:** Top: A scene with a participating medium, illuminated by two single-sided disk lights (invisible to camera rays), rendered with global illumination at  $1280 \times 960$  resolution with 256 samples/pixel. Middle: Just the indirect illumination component. Bottom: Zoom-ins into the indirect component. Even with this high sample count, uniform area sampling produces a noisy image with fireflies in the indirect illumination.