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Practical product sampling for single scattering in media: supplemental document

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1. Introduction

In this supplemental document we provide technical details on our polynomial-expansion-based scheme for importance sampling single scattering from a point source along a ray. We utilize a parameterization where θ is the angle subtended at the point source by the scattering location along the ray (see paper Fig. 1). An order-6 Taylor expansion for a function $f(\theta)$ around $\theta = 0$ has the form

$$\mathcal{T}_f(\theta) = \sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!}\theta^n = \sum_{n=0}^{6} Q_{f,n}\theta^n. \quad (1)$$

The main purpose of this document is to provide the values of the polynomial coefficients $Q_{f,n}$ for $f \in \{T, \rho\}$, i.e. for importance sampling transmittance (Section 2) and phase function (Section 3). In Section 4, we discuss the sampling of the product of such an expansion and our analytical sampling of the point source's cosine foreshortening term (paper Section 3.1).

Knowing the above polynomial coefficients, one can normalize $\mathcal{T}_f(\theta)$ to a PDF with a corresponding CDF, respectively

$$p_f(\theta) = \frac{\mathcal{T}_f(\theta)}{C_f(\theta_{\min},\theta_{\max})}, \qquad P_f(\theta) = \frac{C_f(\theta_{\min},\theta)}{C_f(\theta_{\min},\theta_{\max})}. \quad (2)$$

The normalization factor $C_f(a,b) = \int_a^b \mathcal{T}_f(\theta)\,\mathrm{d}\theta$ is also a polynomial with coefficients obtained by straightforwardly integrating each summand in Eq. (1). The resulting CDF P_f is not analytically invertible, but numerical root finding works particularly well for this type of function (see paper Section 3.2).

2. Taylor expansion of transmittance

In homogeneous media, we can evaluate the transmittance over the entire (two-segment) path as a function of θ (see paper Section 3):

$$T(\theta) = e^{-\mu_t(h(\tan\theta+\sec\theta)+t_h)} = e^{-\mu_t h(\tan\theta+\sec\theta)}e^{-\mu_t t_h}, \quad (3)$$

where h is the (perpendicular) distance between the point source and the ray, and t_h is the distance from the ray origin and the point source's projection onto the ray (see paper Fig. 1). Applying the order-6 Taylor expansion from Eq. (1) with $f \equiv T$ yields the following polynomial coefficients:

$$Q_{T,0} = e^{-\mu_t(D+\Delta)}, \quad (4)$$
$$Q_{T,1} = -u_1, \quad (5)$$
$$Q_{T,2} = \frac{1}{2}(Q_{T,1} + u_2), \quad (6)$$
$$Q_{T,3} = \frac{1}{6}(2Q_{T,1} + 3u_2 - u_3), \quad (7)$$
$$Q_{T,4} = \frac{1}{24}(5Q_{T,1} + 11u_2 - 6u_3 + u_4), \quad (8)$$
$$Q_{T,5} = \frac{1}{120}(16Q_{T,1} + 45u_2 - 35u_3 + 10u_4 - u_5), \quad (9)$$
$$Q_{T,6} = \frac{1}{720}(61Q_{T,1} + 211u_2 - 210u_3 + 85u_4 - 15u_5 + u_6), \quad (10)$$
$$u_n = \mu_t^n D^n Q_{T,0}. \quad (11)$$

3. Taylor expansion of the Henyey-Greenstein phase function

The Henyey-Greenstein model is a (phase) function of the scattering angle that we reparameterize by the θ angle of the single-scattering configuration we use (see paper Fig. 1):

$$\rho(\theta) = \rho_{\mathrm{HG}}\left(\theta + \frac{\pi}{2}\right) = \frac{1}{4\pi}\frac{1-g^2}{(1+g^2+2g\sin\theta)^{3/2}}. \quad (12)$$

Applying the order-6 Taylor expansion from Eq. (1) with $f \equiv \rho$ yields the following polynomial coefficients:

$$Q_{\rho,0} = v_0, \quad (13)$$
$$Q_{\rho,1} = -3v_1, \quad (14)$$
$$Q_{\rho,2} = 15v_2, \quad (15)$$
$$Q_{\rho,3} = \frac{1}{2}(v_1 - 35v_3), \quad (16)$$
$$Q_{\rho,4} = \frac{1}{8}(315v_4 - 20v_2), \quad (17)$$
$$Q_{\rho,5} = \frac{1}{40}(350v_3 - 3465v_5 - v_1), \quad (18)$$
$$Q_{\rho,6} = \frac{1}{48}(9009v_6 - 1260v_4 + 16v_2), \quad (19)$$
$$v_n = \frac{(1-g^2)\,g^n}{4\pi\left(1+g^2\right)^{2n+3}}. \quad (20)$$

4. Analytical Taylor expansion and cosine product

An expansion $\mathcal{T}_f(\theta)$, with $f \in \{T, \rho\}$, can be multiplied with the emitter cosine foreshortening $N(\theta)$, to sample from the product. The corresponding PDF and CDF have forms similar to Eq. (2):

$$p_{N*f}(\theta) = \frac{N(\theta)\,\mathcal{T}_f(\theta)}{C_{N*f}(\theta_{\min},\theta_{\max})}, \quad P_{N*f}(\theta) = \frac{C_{N*f}(\theta_{\min},\theta)}{C_{N*f}(\theta_{\min},\theta_{\max})}. \tag{21}$$

The CDF P_{N*f} again is not analytically invertible, so we sample from it via numerical root finding. The normalization factor $C_{N*f}(a,b) = \int_a^b N(\theta)\mathcal{T}_f(\theta)\,\mathrm{d}\theta$ has a closed-form expression, obtained through integration by parts or a computer algebra system:

$$\begin{aligned} C_{N*f}(a,b) \;=\; & Q_f^{a,b}(b)\cos b - Q_f^{a,b}(a)\cos a + \\ & Q_f^{a,\text{-}b}(b)\sin b - Q_f^{a,\text{-}b}(a)\sin a, \end{aligned} \tag{22}$$

where the sine and cosine factors come from $N(\theta)$. $Q_f^{k,l}$ are polynomials that have the following expressions:

$$Q_f^{k,l}(\theta) = \sum_{n=0}^{6} Q_{f,n}^{k,l}\theta^n, \tag{23}$$

$$\begin{aligned} Q_{f,0}^{k,l} = & -l\,Q_{f,0} + 1k\,Q_{f,1} - 6k\,(Q_{f,3} - 20Q_{f,5}) \\ & - 2l\,(Q_{f,2} - 12Q_{f,4} + 360Q_{f,6}), \end{aligned} \tag{24}$$

$$\begin{aligned} Q_1^{k,l} = & -l\,Q_{f,1} + 2k\,Q_{f,2} - 6l\,(Q_{f,3} - 20Q_{f,5}) \\ & - 24k\,(Q_{f,4} - 30Q_{f,6}), \end{aligned} \tag{25}$$

$$\begin{aligned} Q_2^{k,l} = & -l\,Q_{f,2} + 3k\,Q_{f,3} - 60k\,Q_{f,5} \\ & - 12l\,(Q_{f,4} - 30Q_{f,6}), \end{aligned} \tag{26}$$

$$Q_3^{k,l} = -l\,Q_{f,3} + 4k\,Q_{f,4} - 20l\,Q_{f,5} - 120k\,Q_{f,6}, \tag{27}$$

$$Q_4^{k,l} = -l\,Q_{f,4} + 5k\,Q_{f,5} - 30l\,Q_{f,6}, \tag{28}$$

$$Q_5^{k,l} = -l\,Q_{f,5} + 6k\,Q_{f,6}, \tag{29}$$

$$Q_6^{k,l} = -l\,Q_{f,6}. \tag{30}$$